2. Integriranje pomoću razlaganja podintegralne f-je na dijelove

Ako podintegralna f-ja predstavlja algebarsku sumu nekoliko članova, tada, prema osobini IV

$\left(\int [f_1(x) + f_2(x) - f_3(x)]\, dx = \int f_1(x)\, dx + \int f_2(x)\, dx - \int f_3(x)\, dx \right)$ možemo integrirati svaki član posebno.

Koristeći ovo, mnoge integrale možemo svesti na sumu jednostavnijih integrala.

Odrediti integrale

a) $\int (3x^2 - 2x + 5)\, dx$ b) $\int \frac{2x^2 + x - 1}{x^3}\, dx$ c) $\int (1 + e^x)^2\, dx$

d) $\int \frac{2x + 3}{x^2 - 5}\, dx$ e) $\int \frac{x^2}{x^2 + 1}\, dx$ f) $\int tg^2\varphi\, d\varphi$.

Rj.

a)
$$\int (3x^2 - 2x + 5)\, dx = \int 3x^2\, dx - \int 2x\, dx + \int 5\, dx =$$
$$= 3\int x^2 dx - 2\int x\, dx + 5\int dx = 3\,\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 5x + c$$
$$= x^3 - x^2 + 5x + c$$

b)
$$\int \frac{2x^2 + x - 1}{x^3}\, dx = 2\int \frac{dx}{x} + \int x^{-2} dx - \int x^{-3} dx =$$
$$= 2\ln|x| + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + c = 2\ln|x| - \frac{1}{x} - \frac{1}{2x^2} + c$$

c)
$$\int (1 + e^x)^2\, dx = \int (1 + 2e^x + (e^x)^2)\, dx = \int dx + 2\int e^x dx + \int e^{2x} dx$$
$$= \left| \begin{array}{l} d(2x) = 2\,dx \\ dx = \frac{1}{2}\, d(2x) \end{array} \right| = \int dx + 2\int e^x dx + \frac{1}{2}\int e^{2x} d(2x) =$$
$$= x + 2e^x + \frac{1}{2}\, e^{2x} + c$$

d)
$$\int \frac{2x+3}{x^2-5}\, dx = \int \frac{2x}{x^2-5}\, dx + 3\int \frac{dx}{x^2-5} = \int \frac{d(x^2-5)}{x^2-5} + 3\int \frac{dx}{x^2-5}$$
$$= \ln|x^2 - 5| + \frac{3}{2\sqrt{5}} \ln\left|\frac{x - \sqrt{5}}{x + \sqrt{5}}\right| + c$$

e) $$\int \frac{x^2}{x^2+1}\,dx = \int \frac{(x^2+1)-1}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right) dx =$$

$$= \int dx - \int \frac{dx}{x^2+1} = x - \operatorname{arctg} x + c$$

f) $$\int tg^2\varphi \, d\varphi = \int (tg\varphi)^2 d\varphi = \int \left(\frac{\sin\varphi}{\cos\varphi}\right)^2 d\varphi =$$

$$= \int \frac{\sin^2\varphi}{\cos^2\varphi}\,d\varphi = \int \frac{1-\cos^2\varphi}{\cos^2\varphi}\,d\varphi = \int \left(\frac{1}{\cos^2\varphi} - 1\right) d\varphi$$

$$= tg\varphi - \varphi + c.$$

Zadaci za vježbu

Odrediti integrale

1. $\int \left(2\sqrt[5]{x} - \sqrt[3]{2x} + 5\right) dx$

2. $\int (\sin\varphi - \cos\varphi)^2 \, d\varphi$

3. ** $\int \frac{x^2+1}{x^2-1} \, dx$

4. $\int \frac{5x^2-6x+1}{\sqrt{x}} \, dx$

5. ** $\int \frac{x^3}{x^2+6} \, dx$

6. $\int (\operatorname{tg} x + \operatorname{ctg} x)^2 \, dx$

7. $\int (e^x - e^{-x})^3 \, dx$

8. * $\int \frac{x^2-2}{x+2} \, dx$

* Racionalni algebarski razlomak nazivamo svodljiv, ako je stepen polinoma u brojniku veći ili jednak stepenu polinoma u nazivniku.

** Ovdje, kao u rješenju zadatka $\int \frac{x^2}{x^2+1} dx$, podintegralni svodljiv razlomak napisati u nesvodljivom obliku.

Rj:

1. $\frac{5}{3} x \sqrt[5]{x} - \frac{3\sqrt[3]{2}}{4} x \sqrt[3]{x} + 5x$

2. $\varphi + \frac{1}{2}\cos 2\varphi$

3. $x + \ln\left|\frac{x-1}{x+1}\right|$

4. $2\sqrt{x}\,(x-1)^2$

5. $\frac{x^2}{2} - 3\ln(x^2+6)$

6. $\operatorname{tg} x - \operatorname{ctg} x$

7. $\frac{1}{2}(e^{2x} - e^{-2x}) - 2x$

8. $\frac{(x-2)^2}{2} + 2\ln|x+2|$.

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija pomoću razlaganja podintegralne funkcije na dijelove)

1. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3} dx$

Rj. $\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3} dx = \int \left(\frac{\sqrt{x}}{x^3} - e^x + \frac{1}{x} \right) dx = \int \left(x^{-\frac{5}{2}} - e^x + \frac{1}{x} \right) dx$

$= \frac{x^{-\frac{5}{2}+1}}{-\frac{3}{2}} - e^x + \ln|x| + c = -\frac{2}{3} \cdot \frac{1}{\sqrt{x^3}} - e^x + \ln|x| + C =$

$= C - \frac{2}{3x\sqrt{x}} - e^x + \ln|x| + c$

b) $\int (2x^{-1,2} + 3x^{-0,8} - 5x^{0,38}) dx$

Rj. $\int (2x^{-1,2} + 3x^{-0,8} - 5x^{0,38}) dx = 2 \cdot \frac{x^{-1,2+1}}{-0,2} + 3 \cdot \frac{x^{-0,8+1}}{0,2} - 5 \cdot \frac{x^{0,38+1}}{1,38} + C$

$= -\frac{2}{0,2} x^{-0,2} + \frac{3}{0,2} x^{0,2} - \frac{5}{1,38} x^{1,38} + C = C - 10x^{-0,2} + 15x^{0,2} - 3,62x^{1,38}$

c) $\int \left(\frac{1-z}{z} \right)^2 dz$

Rj. $\int \left(\frac{1-z}{z} \right)^2 dz = \int \left(\frac{1}{z} - 1 \right)^2 dz = \int \left(\frac{1}{z^2} - \frac{2}{z} + 1 \right) dz = \int \left(z^{-2} - 2\frac{1}{z} + 1 \right) dz =$

$= \frac{z^{-1}}{-1} - 2\ln|z| + z + c = C - \frac{1}{z} - \ln z^2 + z$

d) $\int \frac{(1-x)^2}{x\sqrt{x}} dx$

Rj. $\int \frac{(1-x)^2}{x\sqrt{x}} dx = \int \frac{1-2x+x^2}{x^{\frac{3}{2}}} dx = \int \left(x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}} + x^{\frac{1}{2}} \right) dx =$

$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = -2 \cdot \frac{1}{x^{\frac{1}{2}}} - 2 \cdot 2\sqrt{x} + \frac{2}{3}\sqrt{x^3} + C = \frac{-2}{\sqrt{x}} - 4\sqrt{x} + \frac{2}{3} x\sqrt{x} + C$

$= \frac{-2 \cdot 3 - 4\sqrt{x} \cdot 3\sqrt{x} + 2x\sqrt{x} \cdot \sqrt{x}}{3\sqrt{x}} + C = \frac{2x^2 - 12x - 6}{3\sqrt{x}} + C$

e) $\displaystyle\int \frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx$ $\qquad \frac{1}{2}-\frac{1}{3}=\frac{9-2}{18}=\frac{7}{18} \qquad \frac{7}{18}-\frac{4}{18}=\frac{3}{18}=\frac{1}{6} \qquad \frac{12}{18}=\frac{2}{3}$

Rj: $$\int \frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx=\int\left(\frac{1+\sqrt{x}}{x^{\frac{1}{3}}}\right)^3 dx=\int\left(x^{-\frac{1}{3}}+x^{\frac{7}{18}}\right)^3 dx=\int\left(x^{-\frac{1}{3}}+3\cdot x^{-\frac{2}{3}}\cdot x^{\frac{7}{18}}+3x^{-\frac{1}{3}}\cdot x^{\frac{14}{18}}+\right.$$

$$\left.+x^{\frac{7}{6}}\right)dx=\int\left(x^{-\frac{1}{3}}+3x^{\frac{1}{6}}+3x^{\frac{2}{3}}+x^{\frac{7}{6}}\right)dx=\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+3\cdot\frac{x^{\frac{7}{6}}}{\frac{7}{6}}+3\,\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+\frac{x^{\frac{13}{6}}}{\frac{13}{6}}+C$$

$$=\frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}\sqrt[6]{x^7}+\frac{9}{5}\sqrt[3]{x^5}+\frac{6}{13}\sqrt[6]{x^{13}}+C=$$

$$=\frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}x\sqrt[6]{x}+\frac{9}{5}x\sqrt[3]{x^2}+\frac{6}{13}x^2\sqrt[6]{x}+C$$

f) $\displaystyle\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx$ $\qquad \frac{2}{3}-\frac{1}{2}=\frac{4-3}{6}=\frac{1}{6} \qquad \frac{1}{4}-\frac{1}{2}=\frac{1-2}{4}=-\frac{1}{4}$

Rj: $$\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx=\int\frac{x^{\frac{2}{3}}-x^{\frac{1}{4}}}{x^{\frac{1}{2}}}\,dx=\int\left(x^{\frac{1}{6}}-x^{-\frac{1}{4}}\right)dx=\frac{x^{\frac{7}{6}}}{\frac{7}{6}}-\frac{x^{\frac{3}{4}}}{\frac{3}{4}}+C$$

$$=\frac{6}{7}\sqrt[6]{x^7}-\frac{4}{3}\sqrt[4]{x^3}+C=\frac{6}{7}x\sqrt[6]{x}-\frac{4}{3}\sqrt[4]{x^3}+C$$

g) $\displaystyle\int\frac{dx}{\sqrt{3-3x^2}}$

Rj: $$\int\frac{dx}{\sqrt{3-3x^2}}=\int\frac{dx}{\sqrt{3(1-x^2)}}=\int\frac{dx}{\sqrt{3}\cdot\sqrt{1-x^2}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{1-x^2}}=\frac{\sqrt{3}}{3}\arcsin x+C$$

h) $\displaystyle\int\frac{3\cdot2^x-2\cdot3^x}{2^x}\,dx$

Rj: $$\int\frac{3\cdot2^x-2\cdot3^x}{2^x}\,dx=\int\left(3-2\left(\frac{3}{2}\right)^x\right)dx=3x-2\cdot\frac{\left(\frac{3}{2}\right)^x}{\ln\frac{3}{2}}+C=3x-\frac{2\cdot1{,}5^x}{\ln1{,}5}+C$$

i) $\displaystyle\int\frac{1+\cos^2x}{1+\cos2x}\,dx$

Rj: $$\int\frac{1+\cos^2x}{1+\cos2x}\,dx=\int\frac{1+\cos^2x}{\underbrace{\sin^2x+\cos^2x}_{1}+\underbrace{\cos^2x-\sin^2x}_{\cos2x}}\,dx=\int\frac{1+\cos^2x}{2\cos^2x}\,dx$$

$$=\int\left(\frac{1}{2\cos^2x}+\frac{\cos^2x}{2\cos^2x}\right)dx=\frac{1}{2}\int\left(\frac{1}{\cos^2x}+1\right)dx=\frac{1}{2}\operatorname{tg}x+\frac{1}{2}x+C$$

2. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$

Rj. $\int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx =$

$= -\operatorname{ctg} x - \operatorname{tg} x + C = C - \operatorname{ctg} x - \operatorname{tg} x$

b) $\int \operatorname{tg}^2 x \, dx$

Rj. $\int \operatorname{tg}^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1 - \cos^2 x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \operatorname{tg} x - x + C$

c) $\int \operatorname{ctg}^2 x \, dx$

Rj. $\int \operatorname{ctg}^2 x \, dx = \int \left(\frac{\cos x}{\sin x} \right)^2 dx = \int \frac{\cos^2 x}{\sin^2 x} dx = \int \frac{1 - \sin^2 x}{\sin^2 x} dx = \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = C - \operatorname{ctg} x - x$

d) $\int 2 \sin^2 \frac{x}{2} dx$

$$\left. \begin{array}{l} 1 = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \\ \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \end{array} \right\} - \Rightarrow \boxed{1 - \cos x = 2 \sin^2 \frac{x}{2}}$$

Rj. $\int 2 \sin^2 \frac{x}{2} dx = \int (1 - \cos x) dx = x - \sin x + C$

e) $\int \frac{(1 + 2x^2) dx}{x^2 (1 + x^2)}$

Rj. $\int \frac{(1 + 2x^2)}{x^2 (1 + x^2)} dx = \int \frac{1 + x^2 + x^2}{x^2 (1 + x^2)} dx = \int \left[\frac{1 + x^2}{x^2 (1 + x^2)} + \frac{x^2}{x^2 (1 + x^2)} \right] dx = \int \left(\frac{1}{x^2} + \frac{1}{1 + x^2} \right) dx =$

$= \frac{x^{-1}}{-1} + \operatorname{arctg} x + C = C - \frac{1}{x} + \operatorname{arctg} x$

f) $\int \frac{(1 + x)^2 dx}{x (1 + x^2)}$

Rj. $\int \frac{(1 + x)^2 dx}{x (1 + x^2)} = \int \frac{1 + 2x + x^2}{x (1 + x^2)} dx = \int \left(\frac{1 + x^2}{x (1 + x^2)} + \frac{2x}{x (1 + x^2)} \right) dx = \int \left(\frac{1}{x} + \frac{2}{1 + x^2} \right) dx =$

$= \ln|x| + 2 \operatorname{arctg} x + C$

g) $\displaystyle\int \frac{dx}{\cos 2x + \sin^2 x}$

Rj. $\displaystyle\int \frac{dx}{\cos 2x + \sin^2 x} = \int \frac{dx}{\cos^2 x - \sin^2 x + \sin^2 x} = \int \frac{dx}{\cos^2 x} = tg\, x + C$

h) $\displaystyle\int (\arcsin x + \arccos x)\, dx$

Rj. $\displaystyle\int (\arcsin x + \arccos x)\, dx = \int \left(\alpha - \alpha + \frac{\pi}{2}\right) dx = \frac{\pi}{2} x + C$

$\sin\alpha = x$, α je neki ugao

$x \in [-1, 1]$

$\Rightarrow \arcsin x + \arccos x = \frac{\pi}{2}$

$\arcsin x = \alpha$

$\cos\left(-\alpha + \frac{\pi}{2}\right) = \cos(-\alpha)\cos\frac{\pi}{2} - \sin(-\alpha)\sin\frac{\pi}{2} = -\sin(-\alpha) = \sin\alpha$

tj. $\cos\left(-\alpha + \frac{\pi}{2}\right) = \sin\alpha = x \quad \Rightarrow \quad \arccos x = -\alpha + \frac{\pi}{2}$

Odredite sljedeće integrale:

3. $\int(2x^3+5x^2-7x-6)\,dx=\int 2x^3dx+\int 5x^2dx-\int 7x\,dx-\int 6\,dx=$

$=2\int x^3dx+5\int x^2dx-7\int x\,dx-6\int dx=2\cdot\frac{x^4}{4}+5\cdot\frac{x^3}{3}-7\cdot\frac{x^2}{2}-6x+C$

$=\frac{x^4}{2}+\frac{5x^3}{3}-\frac{7x^2}{2}-6x+C$

4. $\int\frac{5x^7+2x^5-x+6}{x^3}\,dx=\int\left(\frac{5x^7}{x^3}+\frac{2x^5}{x^3}-\frac{x}{x^3}+\frac{6}{x^3}\right)dx=$

$5\int x^4dx+2\int x^2dx-\int x^{-2}dx+6\int x^{-3}dx=5\cdot\frac{x^5}{5}+2\cdot\frac{x^3}{3}-\frac{x^{-1}}{-1}+6\cdot\frac{x^{-2}}{-2}+C$

$=x^5+\frac{2x^3}{3}+\frac{1}{x}-3\cdot\frac{1}{x^2}+C$

5. $\int\sqrt{x\sqrt[3]{x\sqrt{x}}}\,dx=\int\sqrt{x\sqrt[3]{\sqrt{x^2\cdot x}}}\,dx=\int\sqrt{x\sqrt[6]{x^3}}\,dx$

$=\int\sqrt{\sqrt[6]{x^6\cdot x^3}}\,dx=\int\sqrt[12]{x^9}\,dx=\int x^{\frac{9}{12}}dx=\int x^{\frac{3}{4}}dx$

$=\frac{x^{\frac{7}{4}}}{\frac{7}{4}}+C=\frac{4}{7}\sqrt[4]{x^7}+C$

6. $\int(x^2+\sqrt{x})^2\,dx=\int(x^4+2x^2\sqrt{x}+x)\,dx=\int x^4dx+2\int \underbrace{x^2\cdot x^{\frac{1}{2}}}_{x^{\frac{5}{2}}}dx+\int x\,dx$

$=\frac{x^5}{5}+2\cdot\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{x^2}{2}+C=\frac{x^5}{5}+\frac{4}{7}\sqrt{x^7}+\frac{x^2}{2}+C$

7. $\int\frac{x}{x+1}\,dx=\int\frac{x+1-1}{x+1}\,dx=\int\left(1-\frac{1}{x+1}\right)dx=$

$=\int dx-\int\frac{dx}{x+1}=x-\ln|x+1|+C$

8. $\int \frac{x^2}{x-1}dx = \int \frac{x^2-1+1}{x-1}dx = \int \frac{x^2-1}{x-1}dx + \int \frac{1}{x-1}dx =$

$= \int \frac{(x-1)(x+1)}{(x-1)}dx + \int \frac{1}{x-1}dx = \int (x+1)dx + \int \frac{1}{x-1}dx = \frac{x^2}{2} + x + \ln|x-1| + C$

9. $\int \frac{x^2}{x+2}dx = \int \frac{x^2+4-4}{x+2}dx = \int \left(\frac{x^2-4}{x+2} + \frac{4}{x+2}\right)dx =$

$= \int \frac{(x-2)(x+2)}{x+2}dx + 4\int \frac{dx}{x+2} = \int (x-2)dx + 4\int \frac{dx}{x+2} = \frac{x^2}{2} - 2x + 4\ln|x+2| + C$

II način bi bio da podjelimo x^2 sa $x+2$ pa izvadimo integral od dobijenog rezultata $x^2 : (x+2) = x - 2 + \frac{4}{x+2}$ $\quad \int \frac{x^2}{x+2}dx = \int \left(x - 2 + \frac{4}{x+2}\right)dx$

10. $\int \frac{x^3}{x-3}dx = \int \frac{x^3-27+27}{x-3}dx = \int \frac{x^3-27}{x-3}dx + \int \frac{27}{x-3}dx =$

$= \int \frac{(x-3)(x^2+3x+9)}{x-3}dx + \int \frac{27}{x-3}dx = \int (x^2+3x+9)dx + 27\int \frac{dx}{x-3}$

$= \frac{x^3}{3} + \frac{3x^2}{2} + 9x + 27\ln|x-3| + C$

11. $\int tg^2x\,dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx - \int dx$

$= tg\,x - x + C$

12. $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}}dx$ $\qquad$ Rj. $-\frac{24}{17}\sqrt[12]{x^{17}} + \frac{4}{5}\sqrt[4]{x^5} + \frac{4}{3}\sqrt[4]{x^3} + C$

13. $\int \frac{e^{3x}+1}{e^x+1}dx$ $\qquad$ Rj. $\frac{1}{2}e^{2x} - e^x + x + C$

14. $\int \frac{1}{\sin^2 2x}dx$ $\qquad$ Rj. $-\frac{1}{2}\cdot\frac{\cos 2x}{\sin 2x} + C$